

A Geometric Proof for the Variation Diminishing Property of B -Spline Approximation

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A geometric proof for the variation diminishing property of B -spline approximation is given. The proof is based primarily upon a generalized form of the de Boor-Cox algorithm and the intuitively obvious fact that piecewise linear interpolation is variation diminishing. Previous proofs [4, 8] employed the mathematical methods of total positivity, a machinery which is available only after reading [8].

1. INTRODUCTION

Let $T := (t_i)_{i=1}^{n+k}$ be a given real nondecreasing sequence with $t_i < t_{i+k}$, all i , and let $(a_i)_{i=1}^n$ be a sequence of real numbers. The B -spline approximation S to the sequence (a_i) is defined by

$$S(t) = \sum_{i=1}^n a_i B_{ik}(t). \quad (1.1)$$

Here

$$B_{ik}(t) = g_k(t_i, \dots, t_{i+k}; t)(t_{i+k} - t_i),$$

where $g_k(t_i, \dots, t_{i+k}; t)$ is the k th divided difference of $g_k(s; t) = (s - t)_+^{k-1}$ as a function of s at the $k + 1$ points t_i, \dots, t_{i+k} and $(s)_+ = \max(s, 0)$.

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Schoenberg [9] noted that B -spline approximation exhibits the following remarkable variation diminishing property:

$$V[S] \leq V[a_1, \dots, a_n], \quad (1.2)$$

where $V[a_1, \dots, a_n]$ is the number of sign changes of the indicated sequence, zero terms discarded, and

$$V[S] = \sup V[S(x_1), S(x_2), \dots, S(x_m)],$$

where the supremum is extended over all nondecreasing sequences $\{x_1, x_2, \dots, x_m\}$, with m an arbitrary integer.

Inequality (1.2) was first proved by Karlin in [8] in the more general setting of Chebyshev splines, and a more specific proof for polynomial splines is given in [4]. However, both these proofs rely specifically on first establishing the total positivity of the relevant spline collocation matrix. The machinery necessary to establish and apply this intermediate result appears to be available only after reading [8]. In this paper we establish (1.2) directly from a recent result (Cohen *et al.* [5]) concerning geometric construction techniques for B -splines.

2. VARIATION DIMINISHING PROPERTY

THEOREM 1 (Cohen *et al.* [4]). *Let S , n , k , T , and $(a_i)_1^n$ be as in (1.1). Let $T' = t'_1, t'_2, \dots, t'_p$ be additional knots. For $m = n + p$ define*

$$X = [x_1, \dots, x_{m+k}] = T \cup T'$$

to be a new knot sequence (called a refinement) again in nondecreasing order. With N_{jk} the B -splines on T' let $(d_j)_1^m$ be the coefficients for S on T' , i.e.,

$$S(t) = \sum_{j=1}^m d_j N_{jk}(t). \quad (2.1)$$

Then for $r = 1, 2, \dots, k$, and $t_u < x_j < t_{u+1}$,

$$d_j = \sum_{i=u-k+r}^u a_{i,j}^{[r]} b_{i,k-r+1}(j), \quad (2.2)$$

where

$$\begin{aligned} a_{i,j}^{[1]} &= a_i, \\ a_{i,j}^{[r+1]} &= (x_{j+k-r} - t_i) a_{i,j}^{[r]} \\ &\quad + (t_{i+k-r} - x_{j+k-r}) a_{i-1,j}^{[r]} / (t_{i+k-r} - t_i) \end{aligned}$$

and

$$\begin{aligned}
 b_{i1}(j) &= 1, & t_i \leq c_j \leq t_{i+1} \\
 &= 0 & \text{otherwise,} \\
 b_{ik}(j) &= ((x_{j+k-1} - t_i) b_{ik}(j) \\
 &+ (t_{i+k} - x_{j+k-1}) b_{i+1,k-1}(j)) / (t_{i+k} - t_i).
 \end{aligned}$$

In particular, for $r = k$, we have $d_j = a_{u,j}^{[k]}$.

Proof. The proof in [6] involves discrete B -splines and is omitted here due to its length. A simple proof exists for the special case of one additional knot [1, 2], which can be applied iteratively to obtain the general result.

Note that, for the particular knot refinement which includes a knot x^* of multiplicity $k - 1$, (2.2) specializes to the de Boor–Cox algorithm [3, 7] for the numerical evaluation of B -splines. Several uses of Theorem 1 as a numerical tool are given in [6].

However, our interest in Theorem 1 is not as a numerical tool. We will show that

$$V[S] \leq V[a_1, \dots, a_n].$$

We have already noted that the $a_{ij}^{[k]}$ coefficients are determined by taking samples from the piecewise linear interpolant to the $a_{ij}^{[k-1]}$. It therefore follows from the definition of V that $V[d_1, \dots, d_m] \leq V[a_1, \dots, a_n]$. Let x_1, \dots, x_p be any increasing sequence. Define a new augmented knot sequence for S by inserting x_i into the existing knot sequence enough times so that the x_i appear with multiplicity $k - 1$. Then $(S(x_1), \dots, S(x_p))$ is a subsequence of the (d_1, \dots, d_m) computed by (2.2) since (2.2) corresponds to the de Boor–Cox algorithm in the case of multiple knots. But then

$$V[S(x_1), \dots, S(x_p)] < V[a_1, \dots, a_n],$$

and we are done.

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